

# Approximations for time-dependent distributions in Markovian fluid models

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## Abstract

In this paper we study the distribution of the level at time  $\theta$  of Markovian fluid queues and Markovian continuous time random walks, the maximum (and minimum) level over  $[0, \theta]$ , and their joint distributions. We approximate  $\theta$  by a random variable  $T$  with Erlang distribution and we use an alternative way, with respect to the usual Laplace transform approach, to compute the distributions. We present probabilistic interpretation of the equations and provide a numerical illustration.

**Keywords:** Markov modulated fluid, Erlangization approximations, Distribution in finite time, Joint distribution

## 1 Introduction

In the literature, Markovian fluid models have been analyzed for many years. One of the first papers appeared in the sixties, for instance with Loynes [8] studying the continuous-time behavior of queues. In the eighties, Markovian fluid models started to be more extensively studied, in particular much work has been dedicated to the study of the stationary distribution, see for instance Rogers [14] and Asmussen [3]. In the present paper, we explore a method to compute time-dependent distributions.

Intuitively, a Markovian fluid model represents the evolution in time of some liquid level in a buffer: taps allow the liquid to flow in and out at different rates.

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The buffer may have a finite or an infinite capacity. The flow is controlled by an underlying Markov chain  $\{\varphi(t) : t \in \mathbb{R}^+\}$  with a finite state space  $\mathcal{S}$ , called the *phase process* : when the Markov chain is in phase  $i \in \mathcal{S}$ , the level of the buffer increases with a constant rate  $c_i$ , if  $c_i$  is strictly positive, or it decreases with the rate  $c_i$ , if  $c_i$  is strictly negative. The *level*  $X(t)$  at time  $t$  may be expressed as follows

$$X(t) = X(0) + \int_0^t c_{\varphi(s)} ds. \quad (1)$$

The process  $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$  is called the *Markovian continuous time random walk* in this paper : it is an unrestricted process and the level may become negative as well as positive. The *Markovian fluid queue* denoted by  $\{(Z(t), \varphi(t)) : t \in \mathbb{R}^+\}$  is related to the random walk in the following way : during the intervals of time when  $Z(t) = 0$  and the rate at time  $t$  is negative, the level remains equal to zero. The level  $Z(t)$  at time  $t$  can be expressed as follows

$$Z(t) = X(t) - \inf_{0 \leq v \leq t} X(v). \quad (2)$$

As the environments remain Markovian in the whole paper, we refer to process  $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$  by the name *random walk* and to the process  $\{(Z(t), \varphi(t)) : t \in \mathbb{R}^+\}$  by the name *fluid queue*.

In this work, we focus on determining the distribution of  $X(\theta)$ , and  $Z(\theta)$ , at a finite time  $\theta > 0$ . In the literature, such time-dependent distributions have been studied using Laplace transforms : Ahn and Ramaswami [1] derive time-dependent distributions of a fluid queue in terms of the transform matrix of the busy period duration, i.e. the matrix  $\Psi(s)$  is such that  $\Psi_{ij}(s) = \mathbb{E}[\exp(-s\tau)1_{\{\varphi(\tau)=j\}} | X(0) = 0, \varphi(0) = i]$  for  $\tau = \inf\{t > 0 : Z(t) = 0\}$ ,  $i \in \mathcal{S}_+$  and  $j \in \mathcal{S}_-$ . Here, we use arguments based on the Erlangization method and so avoid Laplace transform calculations. The idea is to replace the fixed time  $\theta$  by an Erlang-distributed random variable  $T$  such that  $\mathbb{E}[T] = \theta$ . Its advantage is that in so doing, we replace integral equations by linear equations. The Erlangization method has been suggested by Asmussen et al. [4] for ruin problems, Ramaswami et al. [13] determine the return probability to the initial level before the end of the Erlang period and Stanford *et al.* [15] analyze the distribution of the time to ruin.

In addition to the distribution of  $X(T)$ , we compute the joint distribution of  $X(T)$  and the minimum level during  $[0, T]$ . We do the same analysis for the joint distribution of  $X(T)$  and the maximum level during  $[0, T]$ . The Erlangization technique is of particular importance in this case of joint distributions because it replaces integral equations, that we would have to manipulate if the time  $\theta$  would be deterministic, by

linear equations. As a consequence, the resulting probability distributions are more easily computable in the randomized version of time.

We observe that  $X(T)$  has a bilateral phase-type (BPH) distribution, the density function of the BPH has been determined in Ahn and Ramaswami [2]. Here, we go beyond the distribution of the level at time  $T$  as mentioned before and we follow a different approach. We shall discuss about the differences and the similarities between the BPH and the distribution at maturity of the random walk in more details at the end of Section 4.

In the next section, we define precisely the random walk and the fluid queue. In Section 3, we explain the Erlangization method and show how to combine this method with the Markovian fluid models. The equations for  $X(T)$  and  $Z(T)$  are different because of the constraint at level 0 in the second process, and we analyze their time-dependent behavior in two separate sections : Section 4 and 5 respectively. We conclude with an illustrative example in Section 6.

## 2 Fluid models

Consider the random walk  $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$ , where  $X(t)$  is defined in (1), and the fluid queue  $\{(Z(t), \varphi(t)) : t \in \mathbb{R}^+\}$ , where  $Z(t)$  is defined in (2). For both of the fluid models, we assume that the input rate  $c_i$  is different from zero for all  $i \in \mathcal{S} = \{1, \dots, m\}$ . We partition  $\mathcal{S}$  into  $\mathcal{S}_+ \cup \mathcal{S}_-$  with  $\mathcal{S}_+ = \{i \in \mathcal{S} : c_i > 0\}$  and  $\mathcal{S}_- = \{i \in \mathcal{S} : c_i < 0\}$ . Similarly, we define the fluid rate matrix  $C = \text{diag}(c_1, \dots, c_m)$  and partition  $C$  into  $C_+$  and  $C_-$ . The infinitesimal generator of  $\{\varphi(t) : t \in \mathbb{R}^+\}$  is denoted by  $A$  and is written, possibly after permutation of rows and columns, as

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}. \quad (3)$$

We define the joint distribution functions  $F_j(x, t) = \mathbb{P}[X(t) \leq x, \varphi(t) = j]$  for the random walk and the joint distribution functions  $H_j(x, t) = \mathbb{P}[Z(t) \leq x, \varphi(t) = j]$  for the fluid queue. The next theorem gives a differential equation for  $F(\cdot, \cdot)$ . This result is well-known and a proof may be found in Mitra [10].

**Theorem 2.1.** *For all  $x \in \mathbb{R}$ ,  $j \in \mathcal{S}$ , the joint distribution functions  $F_j(x, t)$  are a solution of the system of partial differential equations*

$$\frac{\partial}{\partial t} F_j(x, t) = \sum_{i \in \mathcal{S}} F_i(x, t) A_{ij} - c_j \frac{\partial}{\partial x} F_j(x, t).$$

The joint distribution functions  $H_j(x, t)$  are a solution of the same system for  $x \geq 0$  and  $j \in \mathcal{S}$ .  $\square$

Two pairs of matrices play an important role in the next sections. These are matrices of first return probabilities to the initial level and the infinitesimal generators of monotone records. Denote by  $\tau_+(x)$  and  $\tau_-(x)$  the two first passage times

$$\tau_+(x) = \inf\{t > 0 : X(t) > x\} \quad \text{and} \quad \tau_-(x) = \inf\{t > 0 : X(t) < x\}.$$

We denote by  $\Psi_{ij}$  the probability that, starting from  $(x, i)$  at time 0, with  $x \in \mathbb{R}$  and  $i \in \mathcal{S}_+$ , the random walk returns to level  $x$  in a finite time and does so in phase  $j$ , with  $j \in \mathcal{S}_-$  :

$$\Psi_{ij} = \mathbb{P}[\tau_-(x) < \infty, \varphi(\tau_-(x)) = j | X(0) = x, \varphi(0) = i]. \quad (4)$$

As we assume that the process has spatial homogeneity, the matrix  $\Psi$  of *first return probability from above* does not depend on the level  $x$ . Similarly, the matrix  $\hat{\Psi}$  of *first return probabilities from below* has the components

$$\hat{\Psi}_{ij} = \mathbb{P}[\tau_+(x) < \infty, \varphi(\tau_+(x)) = j | X(0) = x, \varphi(0) = i] \quad (5)$$

where  $i \in \mathcal{S}_-$ ,  $j \in \mathcal{S}_+$ . Note that if  $X(0) = x$  and if the initial phase belongs to  $\mathcal{S}_+$ , then  $\tau_+(x) = 0$ ; if, on the contrary, the initial phase belongs to  $\mathcal{S}_-$ , then  $\tau_-(x) = 0$ . The following theorem is equivalent to Theorem 1 in Rogers [14].

**Theorem 2.2.** *The matrix  $\Psi$  is the minimal nonnegative solution of the Riccati equation*

$$C_+^{-1} A_{+-} + C_+^{-1} A_{++} \Psi + \Psi |C_-|^{-1} A_{--} + \Psi |C_-|^{-1} A_{-+} \Psi = 0.$$

*The matrix  $\hat{\Psi}$  is the minimal nonnegative solution of the Riccati equation*

$$|C_-|^{-1} A_{-+} + |C_-|^{-1} A_{--} \hat{\Psi} + \hat{\Psi} C_+^{-1} A_{++} + \hat{\Psi} C_+^{-1} A_{+-} \hat{\Psi} = 0.$$

$\square$

Define  $\vartheta(x) = \tau_-(-x)$ ,  $R(x) = \varphi(\vartheta(x))$  for  $x \geq 0$ . The process  $\{R(x) : x \geq 0\}$  is Markovian, it corresponds to the phase process observed only during those intervals of time in which  $X(t) = \min_{0 \leq s \leq t} X(s)$ , and is called the process of *downward records*. Its generator, the matrix  $U$ , may be expressed in terms of the matrix of first return probability from above as

$$U = |C_-^{-1}| A_{--} + |C_-^{-1}| A_{-+} \Psi,$$

see for instance da Silva Soares and Latouche [6]. Similarly, we may define the process observed only during those intervals of time in which  $X(t) = \max_{0 \leq s \leq t} X(s)$  and we call it the process of *upward records* : its generator may be written in terms of the matrix of first return probabilities from below as

$$\hat{U} = C_+^{-1} A_{++} + C_+^{-1} A_{+-} \hat{\Psi}.$$

### 3 Erlangization method

The Erlang distribution  $T$  with parameters  $\nu$  and  $L$  may be interpreted as the time until absorption of  $\{\phi(t) : t \in \mathbb{R}^+\}$  a Markov chain with  $L$  transient stages where the process spends an exponential time with mean  $\nu^{-1}$  in each stage, until an absorbing state. The infinitesimal generator of  $\{\phi(t) : t \in \mathbb{R}^+\}$  is given by the matrix of order  $L$

$$N = \begin{bmatrix} -\nu & \nu & & \\ & -\nu & \ddots & \\ & & \ddots & \nu \\ & & & -\nu \end{bmatrix}$$

and  $\phi(t)$  is equal to the number of stages that are completed. The mean of  $T$  is  $L\nu^{-1}$  and to approximate a finite time  $\theta$ , we choose  $\nu = L\theta^{-1}$  for a given  $L$ . The variance of  $T$  is  $\theta^2 L^{-1}$  and decreases to 0 as  $L$  increases to  $\infty$ .

We construct next the *Erlangized random walk*  $\{(X(t), \Phi(t)) : t \in \mathbb{R}^+\}$  with  $\Phi(t) = (\varphi(t), \phi(t))$ :  $\Phi(t) = (i, k)$  means that at time  $t$ , the random walk *phase* is  $i$  and the Erlang *stage* is  $k$ , for  $i \in \mathcal{S}$ ,  $k \in \{0, 1, \dots, L-1\}$ . We may write the infinitesimal generator of the *joint phase-and-stage process*  $\{\Phi(t) : t \in \mathbb{R}^+\}$  as follows:

$$Q = \left[ \begin{array}{cc|c} Q_{++} & Q_{+-} & (-N\mathbf{1}_L) \otimes \mathbf{1}_+ \\ Q_{-+} & Q_{--} & (-N\mathbf{1}_L) \otimes \mathbf{1}_- \\ \hline 0 & 0 & 0 \end{array} \right], \quad (6)$$

where  $Q_{++} = N \otimes I + I \otimes A_{++}$ ,  $Q_{+-} = I \otimes A_{+-}$ ,  $Q_{-+} = I \otimes A_{-+}$ ,  $Q_{--} = N \otimes I + I \otimes A_{--}$ ,  $\otimes$  denotes de Kronecker product, and  $\mathbf{1}_+$ ,  $\mathbf{1}_-$  and  $\mathbf{1}_L$  are a column vectors of ones of size  $|\mathcal{S}_+|$ ,  $|\mathcal{S}_-|$  and  $L$  respectively. The *Erlangized fluid queue*  $\{(Z(t), \Phi(t)) : t \in \mathbb{R}^+\}$  is constructed similarly.

We shall need here matrices of first return probabilities *before the end of the Erlang period*

$$\Psi_{(i,l)(j,n)} = \mathbb{P}[\Phi(\tau_-(0)) = (j, n) | X(0) = 0, \Phi(0) = (i, l)], \quad (7)$$

for  $i \in \mathcal{S}_+$ ,  $j \in \mathcal{S}_-$ , and  $0 \leq l, n \leq L-1$ . Note that  $\Psi_{(\cdot, l)(\cdot, n)}$  does not depend on  $l$  and  $n$  but only on  $n-l$ . It is clear that  $\Psi$  has an upper triangular block structure, and that

$$\Psi = \begin{bmatrix} \Psi^{(0)} & \Psi^{(1)} & \Psi^{(2)} & & \Psi^{(L-1)} \\ 0 & \Psi^{(0)} & \Psi^{(1)} & & \Psi^{(L-2)} \\ \vdots & 0 & \Psi^{(0)} & & \Psi^{(L-3)} \\ & \vdots & 0 & \ddots & \vdots \\ & & \vdots & \ddots & \\ & & & & \Psi^{(0)} \end{bmatrix}, \quad (8)$$

where

$$\Psi_{ij}^{(k)} = \mathbb{P}[\Phi(\tau_-(0)) = (j, k) | X(0) = 0, \Phi(0) = (i, 0)]. \quad (9)$$

The structure of  $\hat{\Psi}$  is similar,

$$\hat{\Psi}_{ij}^{(k)} = \mathbb{P}[\Phi(\tau_+(0)) = (j, k) | X(0) = 0, \Phi(0) = (i, 0)], \quad (10)$$

for  $i \in \mathcal{S}_-$ ,  $j \in \mathcal{S}_+$ ,  $0 \leq k \leq L-1$ .

**Remark 3.1.** In the analysis of fluid models, one of the matrices  $\Psi$  and  $\hat{\Psi}$  defined in (4) and (5), is stochastic, that is  $\Psi \mathbf{1} = \mathbf{1}$ , or  $\hat{\Psi} \mathbf{1} = \mathbf{1}$ , or both. Here, the matrices  $\Psi$  and  $\hat{\Psi}$  defined in (7) and (10) are sub-stochastic, i.e.  $\Psi \mathbf{1} < \mathbf{1}$  and  $\hat{\Psi} \mathbf{1} < \mathbf{1}$ , because they are first return probabilities before  $T$ .

The matrices  $\Psi$  and  $\hat{\Psi}$  are recursively determined as follows.

**Theorem 3.2.** (Ramaswami et al. [13], Thm 4)

(a) The matrix  $\Psi^{(0)}$  is the minimal nonnegative solution of

$$\Psi^{(0)} |C_-|^{-1} A_{-+} \Psi^{(0)} + C_+^{-1} (A_{++} - \nu I) \Psi^{(0)} + \Psi^{(0)} |C_-|^{-1} (A_{--} - \nu I) + C_+^{-1} A_{+-} = 0, \quad (11)$$

and for  $1 \leq k \leq L-1$ ,  $\Psi^{(k)}$  is the solution of the linear system

$$\begin{aligned} & \Psi^{(k)} |C_-|^{-1} (A_{--} - \nu I) + \sum_{n=0}^k \Psi^{(n)} |C_-|^{-1} A_{-+} \Psi^{(k-n)} \\ & + C_+^{-1} (A_{++} - \nu I) \Psi^{(k)} + \nu (C_+^{-1} \Psi^{(k-1)} + \Psi^{(k-1)} |C_-|^{-1}) = 0. \end{aligned}$$

(b) The matrix  $\hat{\Psi}^{(0)}$  is the minimal nonnegative solution of

$$\hat{\Psi}^{(0)} C_+^{-1} A_{+-} \hat{\Psi}^{(0)} + |C_-|^{-1} (A_{--} - \nu I) \hat{\Psi}^{(0)} + \hat{\Psi}^{(0)} C_+^{-1} (A_{++} - \nu I) + |C_-|^{-1} A_{-+} = 0, \quad (12)$$

and for  $1 \leq k \leq L-1$ ,

$$\begin{aligned} & \hat{\Psi}^{(k)} C_+^{-1} (A_{++} - \nu I) + \sum_{n=0}^k \hat{\Psi}^{(n)} C_+^{-1} A_{+-} \hat{\Psi}^{(k-n)} \\ & + |C_-|^{-1} (A_{--} - \nu I) \hat{\Psi}^{(k)} + \nu \left( |C_-|^{-1} \hat{\Psi}^{(k-1)} + \hat{\Psi}^{(k-1)} C_+^{-1} \right) = 0. \end{aligned}$$

□

We may define as in the previous Section, the matrices  $\mathbf{U}$  and  $\hat{\mathbf{U}}$  as the infinitesimal generators of the processes of the monotone records before  $T$ . These matrices are respectively given by

$$\mathbf{U} = (I \otimes |C_-|)^{-1} Q_{--} + (I \otimes |C_-|)^{-1} Q_{-+} \Psi,$$

for the generator of the downward records and by

$$\hat{\mathbf{U}} = (I \otimes |C_+|)^{-1} Q_{++} + (I \otimes |C_+|)^{-1} Q_{+-} \hat{\Psi},$$

for the generator of the upwards records. They have the same block-triangular structure as  $\Psi$  and  $\hat{\Psi}$ ; for instance,

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}^{(0)} & \mathbf{U}^{(1)} & \mathbf{U}^{(2)} & & \mathbf{U}^{(L-1)} \\ 0 & \mathbf{U}^{(0)} & \mathbf{U}^{(1)} & & \mathbf{U}^{(L-2)} \\ \vdots & 0 & \mathbf{U}^{(0)} & & \mathbf{U}^{(L-3)} \\ & \vdots & 0 & \ddots & \vdots \\ & & \vdots & \ddots & \\ & & & & \mathbf{U}^{(0)} \end{bmatrix}. \quad (13)$$

In the next sections, we determine the probability distribution of the level random variable evaluated at time  $T$ , where  $T$  has an Erlang( $L, L\theta^{-1}$ ) distribution. We show in the next lemma that such distribution converge, as  $L \rightarrow \infty$ , to those of the same random variable evaluated at  $\theta$ .

It will be usefull in the sequel to use the notation  $\mathcal{Y}_k$  for the sum

$$\mathcal{Y}_k = \sum_{n=1}^k Y_n \quad (14)$$

where the random variables  $Y_n$ , for  $n \geq 1$ , are i.i.d. exponentially distributed random variables with parameter  $\nu$ . In particular,  $\mathcal{Y}_L = T$ .

We write  $\mathbb{P}_i^a[\cdot]$  to denote the conditional probabilities  $\mathbb{P}[\cdot | X(0) = a, \varphi(0) = i]$  for  $i \in \mathcal{S}$  and  $a \in \mathbb{R}$ . We define the vector of probabilities  $\mathbf{r}(\cdot, \cdot)$  such that

$$r_i(y - a, k) = \mathbb{P}_i^a[X(\mathcal{Y}_k) \leq y], \quad (15)$$

for any  $y \in \mathbb{R}$  and  $k \in \{1, \dots, L\}$ . This is also the probability that  $X(T) \leq y$  given that at time 0, the joint phase-and-stage process is  $(i, L - k)$ , i.e. they are  $k$  stages left before  $T$ .

**Lemma 3.3.** *The distribution of the level reached at time  $T \sim \text{Erl}(L, L\theta^{-1})$ , is such that*

$$\lim_{L \rightarrow \infty} r_i(x, L) = \mathbb{P}_i^0[X(\theta) \leq x],$$

for  $i \in \mathcal{S}$ ,  $x \in \mathbb{R}$ .

*Proof.* We have to show that for any given  $x \in \mathbb{R}$  and for all  $\epsilon > 0$ , there exists  $k_0 \in \mathbb{N}$ , such that for all  $L \geq k_0$ ,

$$|r_i(x, L) - \mathbb{P}_i^0[X(\theta) \leq x]| < \epsilon.$$

To simplify the presentation, define  $F(t) = \mathbb{P}_i^0[X(t) \leq x]$ . As  $F(\cdot)$  is continuous, there exists  $\delta > 0$ , such that

$$|t - \theta| < \delta \Rightarrow |F(t) - F(\theta)| < \frac{\epsilon}{2}. \quad (16)$$

We have

$$|F(T) - F(\theta)| = \left| \int_0^\infty F(t)G(dt) - F(\theta) \right|, \quad (17)$$

where  $G(\cdot)$  is the distribution function of  $T$ ,

$$\begin{aligned} &\leq \int_0^{\theta-\delta} |F(t) - F(\theta)| G(dt) + \int_{\theta-\delta}^{\theta+\delta} |F(t) - F(\theta)| G(dt) \\ &\quad + \int_{\theta+\delta}^\infty |F(t) - F(\theta)| G(dt) \end{aligned} \quad (18)$$



Thanks to (16), the middle term in the right-hand side of (18) is bounded by  $\epsilon \mathbb{P}[T \in [\theta - \delta, \theta + \delta]] / 2 \leq \epsilon/2$ . Define  $k_0 > \frac{2\theta^2}{\epsilon\delta^2}$  fixed. If  $L \geq k_0$ , then the other two terms in (18) together are bounded by

$$\mathbb{P}[|T - \theta| > \delta] < \frac{\theta^2}{L\delta^2},$$

by Chebyshev's inequality. We thus obtain the following inequality

$$|r_i(x, L) - \mathbb{P}_i^0[X(\theta) \leq x]| \leq \frac{\epsilon}{2} + \frac{\theta^2}{L\delta^2} \leq \epsilon,$$

which proves the claim.  $\square$

Convergence of the other marginal probability distributions and the joint distributions studied in this paper may be proved by following the same approach. In the remainder of the paper, we assume that  $L$  is fixed.

## 4 Time-dependent distribution of the random walk

### 4.1 Preliminaries

We consider the fluid model without boundary. As we saw in (15), the computation of  $\mathbf{r}$  does not depend on  $a$  and  $y$  but on their difference only. Therefore, to simplify the writing, we suppose that the initial level is 0. In Theorem 4.1, we give recursive equations for the probabilities of the level being positive or negative at time  $\mathcal{Y}_k$ .

The complementary probability of any probability  $\mathbb{P}[\cdot]$  is denoted by  $\bar{\mathbb{P}}[\cdot]$ , i.e.  $\bar{\mathbb{P}}[\cdot] = 1 - \mathbb{P}[\cdot]$ . Denote by  $\mathbf{h}(k)$  the probability vector that the level is above 0 after an Erlang time period with  $k$  stages, given that the process starts in level 0 in phase  $i \in \mathcal{S}_+$ :

$$h_i(k) = \mathbb{P}_i^0[X(\mathcal{Y}_k) > 0]. \quad (19)$$

Similarly,  $\hat{\mathbf{h}}(k)$  is the probability vector that the level is below 0 after an Erlang time period with  $k$  stages, given that the process starts in phase  $i \in \mathcal{S}_-$ , in level 0 :

$$\hat{h}_i(k) = \mathbb{P}_i^0[X(\mathcal{Y}_k) < 0]. \quad (20)$$

In the proof of next theorem, we use the following terminology : we write that there is a *down-and-up-crossing* when the process, starting in a phase of  $\mathcal{S}_+$  at a given level  $y \in \mathbb{R}$ , returns in a finite time to the same level  $y$  in some phase of  $\mathcal{S}_-$ , crosses that level  $y$ , spends some amount of time below  $y$  and crosses that level again in some phase of  $\mathcal{S}_+$ . Similarly, we define the *up-and-down-crossing*.

**Theorem 4.1.** For  $k = 1$ , one has

$$\mathbf{h}(1) = (I - \Psi^{(0)} \hat{\Psi}^{(0)})^{-1} (\mathbf{1} - \Psi^{(0)} \mathbf{1}), \quad (21)$$

$$\hat{\mathbf{h}}(1) = (I - \hat{\Psi}^{(0)} \Psi^{(0)})^{-1} (\mathbf{1} - \hat{\Psi}^{(0)} \mathbf{1}), \quad (22)$$

and for  $k > 1$ , one has

$$\mathbf{h}(k) = (I - \Psi^{(0)} \hat{\Psi}^{(0)})^{-1} \left( \mathbf{1} - \sum_{n=0}^{k-1} \Psi^{(n)} \mathbf{1} + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \Psi^{(m)} \hat{\Psi}^{(n)} \mathbf{h}(k-m-n) \right), \quad (23)$$

$$\hat{\mathbf{h}}(k) = (I - \hat{\Psi}^{(0)} \Psi^{(0)})^{-1} \left( \mathbf{1} - \sum_{n=0}^{k-1} \hat{\Psi}^{(n)} \mathbf{1} + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \hat{\Psi}^{(m)} \Psi^{(n)} \hat{\mathbf{h}}(k-m-n) \right). \quad (24)$$

*Proof.* For  $k = 1$ , the probability that the level is above the initial level at the end of an exponential period of time is the sum of the following probabilities

$$\mathbf{h}(1) = \mathbf{1} - \Psi^{(0)} \mathbf{1} + \Psi^{(0)} \hat{\Psi}^{(0)} \mathbf{h}(1),$$

Indeed,  $\mathbf{1} - \Psi^{(0)} \mathbf{1}$  is the probability that the process remains above level 0 without interruption until  $\mathcal{Y}_1$  and  $\Psi^{(0)} \hat{\Psi}^{(0)} \mathbf{h}(1)$  is the probability that the process makes an up-and-down-crossing before  $\mathcal{Y}_1$  and that at the end of the period, the level is above 0. As the matrices  $\Psi$  and  $\hat{\Psi}$  are sub-stochastic, the inverse of  $(I - \Psi^{(0)} \hat{\Psi}^{(0)})$  exists and (21) is proved. The proof for (22) is similar.

For  $k > 1$ , we have

$$\mathbf{h}(k) = \left( \mathbf{1} - \sum_{n=0}^{k-1} \Psi^{(n)} \mathbf{1} \right) + \chi^{(k)}, \quad (25)$$

where the first bracket is the probability that the level remains above  $y$  during the whole Erlang time period and  $\chi^{(k)}$  denotes the probability that there is at least one down-and-up-crossing before  $T$ . This term  $\chi^{(k)}$  can be decomposed as follows

$$\chi^{(k)} = \sum_{\substack{0 \leq m, n \\ m+n \leq k-1}} \Psi^{(m)} \hat{\Psi}^{(n)} \mathbf{h}(k-m-n). \quad (26)$$

where the  $(m, n)$ -th term means that a down-crossing occurs during the  $m$ -th stage of the Erlang, with probability  $\Psi^{(m)}$ , with  $m \in \{0, \dots, k-1\}$ ; then the level makes an up-crossing  $n$  stages later with probability  $\hat{\Psi}^{(n)}$  and  $n \in \{0, \dots, k-m-1\}$ ;

finally, there remains  $k - m - n$  stages, so we have to multiply these probabilities by  $\mathbf{h}(k - m - n)$ . We thus obtain  $\mathbf{h}(k)$  by solving the following equation

$$\mathbf{h}(k) = \mathbf{1} - \sum_{n=0}^{k-1} \Psi^{(n)} \mathbf{1} + \sum_{\substack{0 \leq m, n \\ 0 \leq m+n \leq k-1}} \Psi^{(m)} \hat{\Psi}^{(n)} \mathbf{h}(k - m - n). \quad (27)$$

Thus (23) is proved and the proof of (24) may be done similarly.  $\square$

With the next proposition, we show that once  $\mathbf{h}(k)$  is known for all  $k \in \{1, \dots, L\}$ ,  $\hat{\mathbf{h}}(k)$  is easily determined, for any  $k \in \{1, \dots, L\}$ , and vice versa.

**Proposition 4.2.** *One has*

$$\hat{\mathbf{h}}(k) = \mathbf{1} - \sum_{n=0}^{k-1} \hat{\Psi}^{(n)} \mathbf{h}(k - n), \quad (28)$$

$$\mathbf{h}(k) = \mathbf{1} - \sum_{n=0}^{k-1} \Psi^{(n)} \hat{\mathbf{h}}(k - n). \quad (29)$$

for all  $k$ .

*Proof.* In (28), the sum is the probability that the level is positive at  $\mathcal{Y}_k$  given that  $\varphi(0) \in \mathcal{S}_-$ . Equation (29) is also immediate.  $\square$

In what follows, we need to decompose the matrix  $\exp(\mathbf{U}x)$  into sub-blocks, we use the following notation  $\mathbf{W}_x = \exp(\mathbf{U}x)$  for  $x \geq 0$  and using the structure (13) of  $\mathbf{U}$ , we write

$$\mathbf{W}_x = \begin{bmatrix} \mathbf{W}_x^{(0)} & \mathbf{W}_x^{(1)} & \mathbf{W}_x^{(2)} & & \mathbf{W}_x^{(L-1)} \\ 0 & \mathbf{W}_x^{(0)} & \mathbf{W}_x^{(1)} & & \mathbf{W}_x^{(L-2)} \\ \vdots & 0 & \mathbf{W}_x^{(0)} & & \mathbf{W}_x^{(L-3)} \\ & \vdots & 0 & \ddots & \vdots \\ & & \vdots & \ddots & \mathbf{W}_x^{(0)} \end{bmatrix}, \quad (30)$$

The element  $[\mathbf{W}_x^{(n)}]_{uv}$  is the conditional probability that the random walk reaches level 0 in phase  $v \in \mathcal{S}_-$  during the stage  $n$  of the Erlang process, given that the process starts in level  $x$  in phase  $u \in \mathcal{S}_-$ . Observe that  $\mathbf{W}_x^{(0)} = \exp(\mathbf{U}^{(0)}x)$  but the other submatrices have more complex expressions. We discuss in Section 6 how to determine these in actual practice. Similarly, we define the matrix  $\hat{\mathbf{W}}_x = \exp(\hat{\mathbf{U}}x)$ , for  $x \geq 0$ . It is also a block-triangular, block-Toeplitz matrix and we denote by  $\hat{\mathbf{W}}_x^{(n)}$ ,  $n = 0, \dots, L - 1$ , the blocks in the first row.

## 4.2 Probability distributions

Recall the definition (15) of  $\mathbf{r}(\cdot, k)$  as the distribution function of the level at time  $\mathcal{Y}_k$ . We define the minimum and the maximum levels reached during an interval,

$$m(t) = \min\{X(s) : 0 \leq s \leq t\} \quad \text{and} \quad M(t) = \max\{X(s) : 0 \leq s \leq t\}, \quad (31)$$

for  $t \in [0, T]$  and we define their conditional distribution vectors  $\boldsymbol{\eta}(\cdot)$  and  $\boldsymbol{\mu}(\cdot)$  given  $X(0) = 0$ :

$$\eta_i(x, k) = \mathbb{P}_i^0 [m(\mathcal{Y}_k) \leq x] \quad \text{and} \quad \mu_i(x, k) = \mathbb{P}_i^0 [M(\mathcal{Y}_k) \leq x] \quad (32)$$

for  $i \in \mathcal{S}$ . The distributions take different forms according to whether the initial phase is in  $\mathcal{S}_-$  or in  $\mathcal{S}_+$  and we partition all the vectors according to the initial phase in a manner conformant with (3).

**Lemma 4.3.** *The conditional distribution of  $m(\mathcal{Y}_k)$ , given the initial level and the initial phase, is as follows:*

(a) *If  $x < 0$ , then*

$$\boldsymbol{\eta}_+(x, k) = \sum_{n=0}^{k-1} \boldsymbol{\Psi}^{(n)} \boldsymbol{\eta}_-(x, k-n), \quad (33)$$

$$\boldsymbol{\eta}_-(x, k) = \sum_{n=0}^{k-1} \mathbf{W}_{|x|}^{(n)} \mathbf{1}. \quad (34)$$

(b) *If  $x \geq 0$ , then*

$$\boldsymbol{\eta}_+(x, k) = \mathbf{1} \quad \text{and} \quad \boldsymbol{\eta}_-(x, k) = \mathbf{1}. \quad (35)$$

*Proof.* Take  $x < 0$ . If  $i \in \mathcal{S}_-$ , we have

$$\eta_i(x, k) = \mathbb{P}_i^0 [\tau_-(x) \leq \mathcal{Y}_k] = \sum_{n=0}^{k-1} \sum_{j \in \mathcal{S}_-} \mathbb{P}_{ij}^0 [\phi(\tau_-(x)) = n] = \mathbf{e}_i \sum_{n=0}^{k-1} \mathbf{W}_{|x|}^{(n)} \mathbf{1},$$

where  $\mathbf{e}_i$  is the vector with 1 in the  $i$ -th component and zero elsewhere. If  $i \in \mathcal{S}_+$ , the level increases at first and it has to return to the initial level 0 in a phase  $u \in \mathcal{S}_-$  during one of the  $k$  Erlang stages, this is given by  $\boldsymbol{\Psi}_{iu}^{(n)}$  for some  $n \in \{0, \dots, k-1\}$ . Then, starting from a phase in  $\mathcal{S}_-$ , it has to reach below  $x$  during the remaining  $k-n$  exponential steps, which has probability  $\boldsymbol{\eta}_-(x, k-n)$ , which concludes the proof of (33) and (34). The proof of (35) is immediate.  $\square$

**Lemma 4.4.** *The conditional distribution of  $M(\mathcal{Y}_k)$  given the initial level and the initial phase is given as follows.*

(a) *If  $x < 0$ , then*

$$\boldsymbol{\mu}_+(x, k) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}_-(x, k) = \mathbf{0}. \quad (36)$$

(b) *If  $x \geq 0$ , then*

$$\boldsymbol{\mu}_+(x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \hat{\mathbf{W}}_x^{(n)} \mathbf{1}, \quad (37)$$

$$\boldsymbol{\mu}_-(x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \hat{\boldsymbol{\Psi}}^{(n)} \bar{\boldsymbol{\mu}}_+(x, k - n). \quad (38)$$

*Proof.* To analyze the distribution of  $M(\mathcal{Y}_k)$ , we follow an argument similar to the proof of Lemma 4.3 and we determine the complementary probability distribution vector with components  $\bar{\mu}_i(x, k) = \mathbb{P}_i^0 [M(\mathcal{Y}_k) > x]$ . This directly leads to equations (36), (37), (38).  $\square$

**Theorem 4.5.** *The conditional distribution of  $X(\mathcal{Y}_k)$  given that the initial level is 0 and given the initial phase is as follows.*

(a) *If  $x \leq 0$ , then*

$$\mathbf{r}_+(x, k) = \sum_{n=0}^{k-1} \boldsymbol{\Psi}^{(n)} \mathbf{r}_-(x, k - n), \quad (39)$$

$$\mathbf{r}_-(x, k) = \sum_{n=0}^{k-1} \mathbf{W}_{|x|}^{(n)} \hat{\mathbf{h}}(k - n). \quad (40)$$

(b) *If  $x > 0$ , then*

$$\mathbf{r}_+(x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \hat{\mathbf{W}}_x^{(n)} \mathbf{h}(k - n), \quad (41)$$

$$\mathbf{r}_-(x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \hat{\boldsymbol{\Psi}}^{(n)} \bar{\mathbf{r}}_+(x, k - n). \quad (42)$$

*Proof.* Assume that  $x$  is negative and the initial phase is in  $\mathcal{S}_-$ , then to obtain (40), we observe that the level has to reach  $x$  during some stage  $n \in \{0, \dots, k-1\}$ , this happens with the probability  $\mathbf{W}_{|x|}^{(n)}$ . Afterwards, the process has to be below level  $x$  at the end of the remainings  $k-n$  stages left, and this occurs with probability  $\hat{\mathbf{h}}(k-n)$ , given in Theorem 4.1.

If  $i \in \mathcal{S}_+$ , the level has first to return the initial level 0 during some stage  $n \in \{0, \dots, k-1\}$ . Then, the process is in a phase of  $\mathcal{S}_-$  and the argument is the same as before. So (39) is proved.

For  $x > 0$ , to find equations (41) and (42) we determine the probability of the event  $[X(\mathcal{Y}_k) > x]$  given the initial level 0 and initial phase  $i$ , the complement probability of  $r_i(x, k)$ , for which the proof is similar.  $\square$

We may use the Erlangization approach to obtain in a simple manner the joint distribution of  $X(\mathcal{Y}_k)$  and the minimum  $m(\mathcal{Y}_k)$  as well as the joint distribution of  $X(\mathcal{Y}_k)$  and the maximum  $M(\mathcal{Y}_k)$ . We use the notation  $\mathcal{P}^\eta(x, y, k)$  for the vector with components

$$\mathcal{P}_i^\eta(x, y, k) = \mathbb{P}[m(\mathcal{Y}_k) \leq x, X(\mathcal{Y}_k) \leq y | X(0) = 0, \Phi(0) = (i, 0)],$$

and  $\mathcal{P}^\mu(x, y, k)$  for the vector with components

$$\mathcal{P}_i^\mu(x, y, k) = \mathbb{P}[M(\mathcal{Y}_k) \leq x, X(\mathcal{Y}_k) \leq y | X(0) = 0, \Phi(0) = (i, 0)],$$

where  $i \in \mathcal{S}$ ,  $k \in \{1, \dots, L\}$ .

**Theorem 4.6.** *The joint probability distribution of  $m(\mathcal{Y}_k)$  and  $X(\mathcal{Y}_k)$  given the initial level and the initial phase, is as follows.*

(a) *If  $x < 0$ , then*

$$\mathcal{P}_+^\eta(x, y, k) = \sum_{n=0}^{k-1} \Psi^{(n)} \mathcal{P}_-^\eta(x, y, k-n), \quad (43)$$

$$\mathcal{P}_-^\eta(x, y, k) = \sum_{n=0}^{k-1} \mathbf{W}_{|x|}^{(n)} \mathbf{r}_-(y-x, k-n). \quad (44)$$

(b) *If  $x \geq 0$  then*

$$\mathcal{P}_+^\eta(x, y, k) = \mathbf{r}_+(y, k) \quad \text{and} \quad \mathcal{P}_-^\eta(x, y, k) = \mathbf{r}_-(y, k). \quad (45)$$

*Proof.* Firstly, assume that  $x < 0$ , we use the same approach as in the proof of (34): the process has to reach down to level  $x$  with probability  $\mathbf{W}_{|x|}^{(n)}$  and at the end of the remaining  $k - n$  stages, takes a value less than  $y - x$  and so we obtain (44).

If  $\varphi(0) \in \mathcal{S}_+$ , then the level has first to come back to the initial level 0 during some stage  $n \in \{0, \dots, k - 1\}$ , at which time the situation is similar to the case  $\varphi(0) \in \mathcal{S}_-$  with  $k - n$  Erlang stages left; this gives (43).

For  $x \geq 0$ , given that the initial level is 0,  $m(\mathcal{Y}_k) \leq x$  with probability one and (45) immediately follows.  $\square$

**Theorem 4.7.** *The joint probability distribution of  $M(\mathcal{Y}_k)$  and  $X(\mathcal{Y}_k)$  given the initial level and the initial phase, is as follows.*

(a) *If  $0 \leq x$  and  $y < x$ ,*

$$\mathcal{P}_+^\mu(x, y, k) = \mathbf{r}_+(y, k) - \sum_{n=0}^{k-1} \hat{\mathbf{W}}_x^{(n)} \mathbf{r}_+(y - x, k - n), \quad (46)$$

$$\mathcal{P}_-^\mu(x, y, k) = \mathbf{r}_-(y, k) - \sum_{\substack{0 \leq m, n \\ m+n \leq k-1}} \hat{\Psi}^{(n)} \hat{\mathbf{W}}_x^{(m)} \mathbf{r}_+(y - x, k - n - m). \quad (47)$$

(b) *If  $0 > x$  or  $y > x$ , then*

$$\mathcal{P}_+^\mu(x, y, k) = \boldsymbol{\mu}_+(x, k) \quad \text{and} \quad \mathcal{P}_-^\mu(x, y, k) = \boldsymbol{\mu}_-(x, k). \quad (48)$$

*Proof.* To prove (46) and (47), we write

$$[X(\mathcal{Y}_k) \leq y, M(\mathcal{Y}_k) \leq x] = [X(\mathcal{Y}_k) \leq y] \setminus [X(\mathcal{Y}_k) \leq y, M(\mathcal{Y}_k) > x],$$

and follow an argument similar to Theorem 4.6. To see (48) is obvious.  $\square$

**Remark 4.8. Link with the bilateral phase-type distribution**

It must be observed that  $X(T)$  is a particular BPH distribution which we briefly define. Suppose that  $\{\zeta(t) : t \in \mathbb{R}^+\}$  a Markov process is defined on the state space  $\mathcal{E} = \{0, 1, \dots, m\}$  where states  $1, \dots, m$  are transient and state 0 is absorbing. The infinitesimal generator is  $G$  with the following structure

$$G = \left[ \begin{array}{c|c} D & \mathbf{d} \\ \hline 0 & 0 \end{array} \right],$$

where  $D$  is a square matrix of order  $m$ ,  $\mathbf{d}$  is a column vector of size  $m$  and  $\mathbf{d} = -D\mathbf{1}$ . The distribution of time  $\Delta$  until absorption, defined by

$$\Delta = \inf \{t \geq 0 : D(t) = 0\},$$

is the *phase-type distribution* with representation  $(\gamma, D)$ , where  $\gamma$  is the initial distribution of  $\{\zeta(t)\}$  over the transient states. Define a *Markov modulated fluid model*  $\{(Y(t), \zeta(t)) : t \in \mathbb{R}^+\}$  such that  $Y(0) = 0$  and  $Y(\cdot)$  varies linearly as follows

$$Y(t) = \int_0^t e_{\zeta(s)} ds,$$

where  $e_i > 0$  if  $i \in \mathcal{E}_+$  and  $e_i < 0$  if  $i \in \mathcal{E}_-$ , and  $\mathcal{E}_+ \cup \mathcal{E}_- \cup \{0\} = \mathcal{E}$ . The distribution of  $Y(\Delta)$  is the *bilateral phase-type (BPH) distribution* with representation  $(\gamma, D, E)$ , where  $E = \text{diag}(e_1, \dots, e_m)$ . It is clear that  $X(T)$  has a BPH distribution where  $G$  is given by (6). The density function  $f$  of  $Y(\Delta)$  may be written as follows, where  $\Psi$ ,  $U$  and  $\mathbf{h}$  are as in (8), (13) and (19), respectively and similarly for  $\hat{\Psi}$ ,  $\hat{U}$  and  $\hat{\mathbf{h}}$ . We suppose that  $\gamma_0 = 0$  so that the initial state is not the absorbing state 0.

**Theorem 4.9.** *The density function  $f$  of  $Y(\Delta)$  is given as follows*

$$f(x) = \begin{cases} \hat{\mathbf{p}} \exp(U|x|)(-U)\hat{\mathbf{h}}, & x < 0, \\ \mathbf{p} \exp(\hat{U}x)(-\hat{U})\mathbf{h}, & x > 0, \end{cases} \quad (49)$$

$$(50)$$

where  $\hat{\mathbf{p}} = \gamma_+ \Psi + \gamma_-$  and  $\mathbf{p} = \gamma_+ + \gamma_- \hat{\Psi}$ .

*Proof.* Take  $x < 0$ . By a similar argument as in Theorem 4.5, we write  $\mathbb{P}[Y(\Delta) \leq x] = \gamma_+ \Psi \exp(U|x|)\hat{\mathbf{h}}$ , if the initial state is in  $\mathcal{E}_+$  and  $\mathbb{P}[Y(\Delta) \leq x] = \gamma_- \exp(U|x|)\hat{\mathbf{h}}$ , if the initial state is in  $\mathcal{E}_-$ . Thus,

$$f(x) = \frac{d}{dx} \left( (\gamma_+ \Psi + \gamma_-) \exp(U|x|)\hat{\mathbf{h}} \right) = \hat{\mathbf{p}} \exp(U|x|)(-U)\hat{\mathbf{h}},$$

and this shows (49). A similar argument holds for the proof of (50).  $\square$

This expression is equivalent to the one given in Theorem 4.1 in Ahn and Ramaswami [2]. To see this, we take the special case  $L = 1$  in order to simplify the presentation. Note that  $\Psi^{(0)}$  defined in (11) and  $\Psi$  defined here are now identical. For  $x > 0$ ,

$$\mathbb{P}[Y(\Delta) > x] = \mathbf{p} \exp(\hat{U}x)(I - \Psi\hat{\Psi})^{-1}(\mathbf{1} - \Psi\mathbf{1}), \quad (51)$$



by (41), (42) and (21). The calculations in Govorun *et al.* [7], page 83 indicate that  $(I - \Psi\hat{\Psi})\hat{U} = K(I - \Psi\hat{\Psi})$  and here the matrix  $(I - \Psi\hat{\Psi})$  is non-singular so that

$$\exp(\hat{U}x)(I - \Psi\hat{\Psi})^{-1} = (I - \Psi\hat{\Psi})^{-1}\exp(Kx)$$

and we may rewrite (51) as

$$\mathbb{P}[Y(\Delta) > x] = \hat{\mathbf{p}}(I - \hat{\Psi}\Psi)^{-1}\exp(Kx)(\mathbf{1} - \hat{\Psi}\mathbf{1}).$$

By post-multiplying the equation (11) for  $\Psi^{(0)}$  by  $\mathbf{1}$ ,

$$E_+^{-1}D_{++}\Psi\mathbf{1} + \Psi|E_-|^{-1}D_{-+}\Psi\mathbf{1} = -(E_+^{-1}D_{+-}\mathbf{1} + \Psi|E_-|^{-1}D_{--}\mathbf{1}).$$

and since  $\mathbf{d} = -D\mathbf{1}$ , after algebraic manipulation, we find

$$(\mathbf{1} - \Psi\mathbf{1}) = (-K)^{-1}[E_+^{-1}\mathbf{d}_+ + \Psi E_-^{-1}\mathbf{d}_-],$$

where  $K = E_+^{-1}D_{++} + \Psi|E_-|^{-1}D_{-+}$ . Thus, we obtain

$$\mathbb{P}[Y(\Delta) > x] = \mathbf{p}(I - \hat{\Psi}\Psi)^{-1}\exp(Kx)(-K)^{-1}[E_+^{-1}\mathbf{d}_+ + \Psi E_-^{-1}\mathbf{d}_-],$$

and

$$f(x) = \mathbf{p}(I - \hat{\Psi}\Psi)^{-1}\exp(Kx)[E_+^{-1}\mathbf{d}_+ + \Psi E_-^{-1}\mathbf{d}_-], \quad \text{if } x > 0. \quad (52)$$

With a similar argument, one shows that

$$f(x) = \hat{\mathbf{p}}(I - \Psi\hat{\Psi})^{-1}\exp(\hat{K}|x|)[\hat{\Psi}E_+^{-1}\mathbf{d}_+ + E_-^{-1}\mathbf{d}_-], \quad \text{if } x < 0. \quad (53)$$

where  $\hat{K} = |E_-|^{-1}D_{--} + E_+^{-1}\hat{\Psi}D_{+-}$ . Equations (52) and (53) correspond to the density function given in Theorem 4.1 in Ahn and Ramaswami [2].

## 5 Time-dependent distribution of fluid queues

### 5.1 Boundary effect

Here, we assume that there is a boundary at the level 0, and we consider the fluid queue  $\{(Z(t), \Phi(t)) : t \in \mathbb{R}^+\}$  where  $Z(t)$  is given by (2). To begin with, we analyze the length of time intervals of time during which the fluid queue remains in level 0, once it gets there. Specifically, we determine the probability that the fluid queue

leaves level 0 during the Erlang stage  $m$ , in a phase  $j \in \mathcal{S}_+$ , given that the process starts in level 0 in phase  $i \in \mathcal{S}_-$ , that probability is denoted by  $\Upsilon_{ij}^{(m)}$ , i.e.

$$\Upsilon_{ij}^{(m)} = \mathbb{P}_{ij}^0[\phi(\beta) = m], \quad (54)$$

where  $\beta = \inf\{t > 0 : \varphi(t) \in \mathcal{S}_+\}$ .

**Theorem 5.1.** *The transition probability matrix of the stages upon leaving level 0, with components defined in (54), is given by*

$$\Upsilon^{(m)} = \nu^m (\nu I - A_{--})^{-(m+1)} A_{-+}. \quad (55)$$

*Proof.* We obtain as follows the probability that, starting from level 0 and a phase in  $\mathcal{S}_-$ ,  $Z(t)$  remains in level 0 for  $m$  exponential periods, and then leaves level 0 in a phase  $j$  of  $\mathcal{S}_+$  before the start of the  $m + 1$ -st stage,

$$\Upsilon^{(m)} = \int_0^\infty e^{A_{--}u} A_{-+} \mathbb{P}[\mathcal{Y}_m < u < \mathcal{Y}_{m+1}] du,$$

where  $\mathcal{Y}_m$  is defined in (14),

$$\begin{aligned} &= \int_0^\infty e^{A_{--}u} A_{-+} \frac{(\nu u)^m}{m!} e^{-\nu u} du, \\ &= \nu^m (\nu I - A_{--})^{-(m+1)} A_{-+}. \end{aligned}$$

□

## 5.2 Distribution at time $T$

In order to determine the distribution of the level at time  $T$ , we need the distribution under taboo of level 0. This way of analysis is standard in the matrix analytic literature for fluid models. For instance, see Ramaswami [12].

The probability that the fluid queue is below level  $x > 0$  at time  $\mathcal{Y}_k$ , under taboo of level 0, given that the process starts in level  $a \in \mathbb{R}^+$ , in phase  $i$ , at time 0, is denoted by

$$g_i(a, x, k) = \mathbb{P}^a[Z(\mathcal{Y}_k) < x, Z(t) > 0, \forall t \in (0, \mathcal{Y}_k] | \Phi(0) = (i, 0)].$$

Here we need to keep track of the initial level  $Z(0) = a$  because of the barrier in level 0:  $\mathbf{g}(a, x, k)$  is not equal to  $\mathbf{g}(0, x - a, k)$  in general.

**Lemma 5.2.** *The probability that the fluid queue is below level  $x > 0$  at time  $\mathcal{Y}_k$ , without returning to level 0, given the initial level  $a \geq 0$  and the initial phase, is*

$$\mathbf{g}_+(a, x, k) = \mathbf{r}_+(x - a, k) - \sum_{\substack{0 \leq m, n \\ m+n \leq k-1}} \Psi^{(m)} \mathbf{W}_a^{(n)} \mathbf{r}_-(x, k - m - n), \quad (56)$$

$$\mathbf{g}_-(a, x, k) = \mathbf{r}_-(x - a, k) - \sum_{n=0}^{k-1} \mathbf{W}_a^{(n)} \mathbf{r}_-(x, k - n), \quad (57)$$

where  $\mathbf{r}(\cdot, \cdot)$  is given in Theorem 4.5.

*Proof.* To prove this, we use the event decomposition

$$[Z(\mathcal{Y}_k) < x, Z(t) > 0, \forall t \in (0, \mathcal{Y}_k]] = [X(\mathcal{Y}_k) < x] \setminus [X(\mathcal{Y}_k) < x, \tau_-(0) < \mathcal{Y}_k],$$

and follow an argument similar to the proof in Theorem 4.6.  $\square$

The vector  $\mathbf{q}(a, x, k)$  of conditional distribution at time  $\mathcal{Y}_k$  is defined like  $\mathbf{g}$ , with the difference that there is no constraint on  $Z(t)$  during the interval  $[0, \mathcal{Y}_k]$ :

$$q_i(a, x, k) = \mathbb{P}_i^a [Z(\mathcal{Y}_k) \leq x]. \quad (58)$$

We give recursive equations for the computation of this vector.

**Theorem 5.3.**

(a) For  $k \geq 1$  and  $x > 0$ , one has

$$\begin{aligned} \mathbf{q}_+(0, x, k) &= (I - \Psi^{(0)} \Upsilon^{(0)})^{-1} \left( \mathbf{g}_+(0, x, k) \right. \\ &\quad \left. + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \Psi^{(n)} \Upsilon^{(m)} \mathbf{q}_+(0, x, k - n - m) \right), \end{aligned} \quad (59)$$

$$\mathbf{q}_-(0, x, k) = \sum_{n=0}^{k-1} \Upsilon^{(n)} \mathbf{q}_+(0, x, k - n), \quad (60)$$

where  $\mathbf{g}_+(0, x, \cdot)$  is given in Lemma 5.2.

(b) For  $k \geq 1$ , for  $a, x > 0$ , one has

$$\mathbf{q}_+(a, x, k) = \mathbf{g}_+(a, x, k) + \sum_{\substack{0 \leq m, n \\ m+n \leq k-1}} \Psi^{(n)} \mathbf{W}_a^{(m)} \mathbf{q}_-(0, x, k - n - m), \quad (61)$$

$$\mathbf{q}_-(a, x, k) = \mathbf{g}_-(a, x, k) + \sum_{n=0}^{k-1} \mathbf{W}_a^{(n)} \mathbf{q}_-(0, x, k - n). \quad (62)$$

*Proof.* There are two ways to be in  $[0, x]$  at time  $\mathcal{Y}_k$ , starting from 0 in a phase of  $\mathcal{S}_+$ : either this occurs without returning to the level 0 during the period  $(0, \mathcal{Y}_k]$ , or the process first comes back to 0, stays there for some amount of time, leaves level 0 and is in  $[0, x]$  at the end of the remaining number of stages. This gives the decomposition

$$\mathbf{q}_+(0, x, k) = \mathbf{g}_+(0, x, k) + \sum_{n=0}^{k-1} \sum_{m=0}^{k-n-1} \Psi^{(n)} \Upsilon^{(m)} \mathbf{q}_+(0, x, k - n - m). \quad (63)$$

which leads to the equation (59).

Equation (60) is obtained by noting that the process first remain in the level 0 for  $n$  Erlang stages and it leaves level 0 in a phase of  $\mathcal{S}_+$ .

We deduce easily the general case for the distribution  $\mathbf{q}(a, x, k)$ , for  $a > 0$ : we decompose this into the probability that  $X(\mathcal{Y}_k) \leq x$  without returning to 0 and the probability that the process returns to 0 before  $\mathcal{Y}_k$ .  $\square$

Note that when  $k = 1$ , the equations simplify. In particular, for instance, the second term in the big bracket in (59) disappear as the sum is empty.

### 5.3 Distributions of the minimum and the maximum

Define the minimum and the maximum level reached during an interval for the model bounded at 0:

$$m_0(t) = \min\{Z(v) : 0 \leq v \leq t\} \quad \text{and} \quad M_0(t) = \max\{Z(v) : 0 \leq v \leq t\}. \quad (64)$$

We denote by  $\boldsymbol{\rho}(a, x, k)$  and by  $\boldsymbol{\delta}(a, x, k)$  the vectors of the conditional distribution function of the minimum and the maximum reached during the Erlang period:

$$\rho_i(a, x, k) = \mathbb{P}_i^a [m_0(\mathcal{Y}_k) \leq x] \quad \text{and} \quad \delta_i(a, x, k) = \mathbb{P}_i^a [M_0(\mathcal{Y}_k) \leq x] \quad (65)$$

for  $i \in \mathcal{S}$ .

**Lemma 5.4.** *The conditional distribution of the minimum level reached by the fluid queue during the Erlang horizon period, given the initial level  $a$  and the initial phase, is*

$$\boldsymbol{\rho}_+(a, x, k) = \boldsymbol{\eta}_+(a - x, k) \quad \text{and} \quad \boldsymbol{\rho}_-(a, x, k) = \boldsymbol{\eta}_-(a - x, k), \quad (66)$$

for  $a, x \geq 0$  and where  $\boldsymbol{\eta}(\cdot, \cdot)$  is given in the Lemma 4.3.

*Proof.* The proof is immediate as it is clear that for  $x \geq 0$ , the event  $[m(t) > x]$  and  $[m_0(t) > x]$  are identical.  $\square$

The analysis of  $M_0(\mathcal{Y}_k)$  is more involved, due to the barrier at zero, and we need a new set of first passage first passage probability matrices between the levels 0 and  $x$ . Denote by  $\Lambda_x^{(k)}$  the matrix with components

$$(\Lambda_x^{(k)})_{ij} = \mathbb{P}_i^0[\tau_+(x) < \tau_-(0), \tau_+(x) < \mathcal{Y}_k, \Phi(\tau_+(x)) = (j, k)],$$

for  $i, j \in \mathcal{S}_+$ ,  $0 \leq k \leq L-1$ , that is  $(\Lambda_x^{(k)})_{ij}$  is the probability, starting from  $(0, i)$ , of reaching level  $x > 0$  in phase  $j$ , during the stage  $k$  of the Erlang, under taboo of level zero. Symmetrically, denote by  $\Psi_x^{(k)}$  the matrix with components

$$(\Psi_x^{(k)})_{iu} = \mathbb{P}_i^0[\tau_-(0) < \tau_+(x), \tau_-(0) < \mathcal{Y}_k, \Phi(\tau_-(0)) = (u, k)]$$

with  $i \in \mathcal{S}_+$ ,  $u \in \mathcal{S}_-$ ,  $0 \leq k \leq L-1$ , that is  $(\Psi_x^{(k)})_{ij}$  is the probability, starting from  $(0, i)$ , of returning to the level 0 during the stage  $k$  of the Erlang without having reached level  $x$ .

Furthermore, define the matrices  $\hat{\Lambda}_x^{(k)}$  and  $\hat{\Psi}_x^{(k)}$  with components

$$(\hat{\Lambda}_x^{(k)})_{ij} = \mathbb{P}_i^x[\tau_-(0) < \tau_+(x), \tau_-(0) < \mathcal{Y}_k, \Phi(\tau_-(0)) = (j, k)],$$

$$(\hat{\Psi}_x^{(k)})_{iu} = \mathbb{P}_i^x[\tau_+(x) < \tau_-(0), \tau_+(x) < \mathcal{Y}_k, \Phi(\tau_+(x)) = (u, k)],$$

with  $i, j \in \mathcal{S}_-$ ,  $u \in \mathcal{S}_+$ ,  $0 \leq k \leq L-1$ .

We denote by  $\Omega_x = \Psi^{(0)}W_x^{(0)}$ ,  $\Omega_x^{(n,k)} = \Psi^{(n)}W_x^{(k-n)}$ ,  $\hat{\Omega}_x = \hat{\Psi}^{(0)}\hat{W}_x^{(0)}$ , and  $\hat{\Omega}_x^{(n,k)} = \hat{\Psi}^{(n)}\hat{W}_x^{(k-n)}$ . The matrices  $\Lambda_x^{(k)}$  and  $\hat{\Lambda}_x^{(k)}$  can be expressed according to the matrices  $\Psi_x^{(k)}$  and  $\hat{\Psi}_x^{(k)}$  which are computed recursively as follows.

**Theorem 5.5.** *Let  $x > 0$ .*

(a) *The matrices  $\Lambda_x^{(0)}$  and  $\hat{\Lambda}_x^{(0)}$  are given by*

$$\Lambda_x^{(0)} = \hat{W}_x^{(0)} - \Psi_x^{(0)}\hat{\Omega}_x \tag{67}$$

$$\hat{\Lambda}_x^{(0)} = W_x^{(0)} - \hat{\Psi}_x^{(0)}\Omega_x \tag{68}$$

where

$$\Psi_x^{(0)} = \left( \Psi^{(0)} - \hat{W}_x^{(0)}\Omega_x \right) (I - \hat{\Omega}_x\Omega_x)^{-1}, \tag{69}$$

$$\hat{\Psi}_x^{(0)} = \left( \hat{\Psi}^{(0)} - W_x^{(0)}\hat{\Omega}_x \right) (I - \Omega_x\hat{\Omega}_x)^{-1}. \tag{70}$$

(b) For  $1 \leq k \leq L-1$ , the matrices  $\Lambda_x^{(k)}$  and  $\hat{\Lambda}_x^{(k)}$  are given by

$$\Lambda_x^{(k)} = \hat{W}_x^{(k)} - \sum_{\substack{0 \leq m, n \\ 0 \leq m+n \leq k}} \Psi_x^{(m)} \hat{\Omega}_x^{(n, k-m)} \quad (71)$$

$$\hat{\Lambda}_x^{(k)} = W_x^{(k)} - \sum_{\substack{0 \leq m, n \\ 0 \leq m+n \leq k}} \hat{\Psi}_x^{(m)} \Omega_x^{(n, k-m)} \quad (72)$$

where the matrices  $\Psi_x^{(k)}$  and  $\hat{\Psi}_x^{(k)}$  are given by

$$\begin{aligned} \Psi_x^{(k)} = & \left( \Psi^{(k)} - \sum_{\substack{0 \leq m \leq k-1 \\ 0 \leq n \leq k-m}} \Lambda_x^{(m)} \Omega_x^{(n, k-m)} \right. \\ & \left. - \left[ \hat{W}_x^{(k)} + \sum_{\substack{0 \leq m \leq k-1 \\ 0 \leq n \leq k-m}} \Psi_x^{(m)} \hat{\Omega}_x^{(n, k-m)} \right] \Omega_x \right) (I - \hat{\Omega}_x \Omega_x)^{-1} \quad (73) \end{aligned}$$

$$\begin{aligned} \hat{\Psi}_x^{(k)} = & \left( \hat{\Psi}^{(k)} - \sum_{\substack{0 \leq m \leq k-1 \\ 0 \leq n \leq k-m}} \hat{\Lambda}_x^{(m)} \hat{\Omega}_x^{(n, k-m)} \right. \\ & \left. - \left[ W_x^{(k)} + \sum_{\substack{0 \leq m \leq k-1 \\ 0 \leq n \leq k-m}} \hat{\Psi}_x^{(m)} \Omega_x^{(n, k-m)} \right] \hat{\Omega}_x \right) (I - \Omega_x \hat{\Omega}_x)^{-1} \quad (74) \end{aligned}$$

*Proof.* The matrix  $\hat{W}_x^{(0)}$  can be decomposed as the sum of the probability that the process reaches level  $x$  before returning to the level 0 and the probability that the process returns to the level 0 before reaching the level  $x$ . It can thus be written as

$$\hat{W}_x^{(0)} = \Lambda_x^{(0)} + \Psi_x^{(0)} \hat{\Psi}_x^{(0)} \hat{W}_x^{(0)},$$

and therefore, equation (67) is proved.

Recall that the matrix  $\Psi^{(0)}$  is such that each component  $\Psi_{ij}^{(0)}$  is the probability, starting from  $(0, i)$ , of returning to the level 0 before one exponential stage is finished, with  $i \in \mathcal{S}_+$  and  $j \in \mathcal{S}_-$ : it can thus be decomposed as the sum

$$\Psi^{(0)} = \Psi_x^{(0)} + \Lambda_x^{(0)} \Psi^{(0)} W_x^{(0)}, \quad (75)$$

where the first term is the probability that the process returns to the level 0 before crossing the level  $x$ , and the second is the probability that the process first visit the

level  $x$  before it returns to the level 0. In the second case, the process must return to level  $x$  in a phase of  $\mathcal{S}_-$  which justifies the factor  $\Psi^{(0)}$ , before eventually going down to level 0. Using (67) in the equation (75) leads to (69).

We prove (68) and (70) in a similar manner. For  $1 \leq k \leq L-1$ , we need to keep track of the stages of the Erlang and readily obtain the expressions (71) and (72) for the matrices  $\Lambda_x^{(k)}$  and  $\hat{\Lambda}_x^{(k)}$ .

The matrix  $\Psi_x^{(k)}$  is obtained by noting that

$$\Psi^{(k)} = \Psi_x^{(k)} + \sum_{\substack{0 \leq m, n \\ 0 \leq m+n \leq k}} \Lambda_x^{(m)} \Psi^{(n)} W_x^{(k-n-m)}, \quad (76)$$

and by replacing  $\Lambda_x^{(m)}$  by its expression given in (71). By reorganizing the terms we find the equation (73).  $\square$

Now we can give the distribution of the maximum level reached during the Erlang interval, given the initial level, we follow the (by now) familiar accounting of the stages of the Erlang interval.

**Theorem 5.6.** (a) Take  $a = 0$ . For  $k \geq 1$  and  $x \geq 0$ ,

$$\begin{aligned} \bar{\delta}_+(0, x, k) &= \left( I - \Psi_x^{(0)} \Upsilon^{(0)} \right)^{-1} \\ &\quad \left( \sum_{n=0}^{k-1} \Lambda_x^{(n)} \mathbf{1} + \sum_{\substack{0 \leq n, l \\ 1 \leq n+l \leq k-1}} \Psi_x^{(n)} \Upsilon^{(l)} \bar{\delta}_+(0, x, k-n-l) \right). \end{aligned} \quad (77)$$

For  $k \geq 1$ , the conditional distribution of  $M_0(\mathcal{Y}_k)$  given the initial level  $a = 0$ , is given by

$$\delta_+(0, x, k) = \mathbf{1} - \bar{\delta}_+(0, x, k), \quad (78)$$

$$\delta_-(0, x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \Upsilon^{(n)} \bar{\delta}_+(0, x, k-n) \quad (79)$$

(b) Take  $a > 0$ . For  $k \geq 1$ ,

$$\begin{aligned} \bar{\delta}_+(a, x, k) = & \left( I - \Psi_{x-a}^{(0)} \hat{\Psi}_a^{(0)} \right)^{-1} \\ & \left[ \sum_{n=0}^{k-1} \Lambda_{x-a}^{(n)} \mathbf{1} + \Psi_{x-a}^{(0)} \hat{\Lambda}_a^{(0)} \bar{\delta}_-(0, x, k) \right. \\ & + \sum_{\substack{0 \leq n, l \\ 1 \leq n+l \leq k-1}} \Psi_{x-a}^{(n)} \left( \hat{\Psi}_a^{(l)} \bar{\delta}_+(a, x, k-n-l) \right. \\ & \left. \left. + \hat{\Lambda}_a^{(l)} \bar{\delta}_-(0, x, k-n-l) \right) \right]. \end{aligned} \quad (80)$$

For  $k \geq 1$ , the conditional distribution of  $M_0(\mathcal{Y}_k)$  given the initial level  $a > 0$ , is given for  $x \geq a$ , by

$$\delta_+(a, x, k) = \mathbf{1} - \bar{\delta}_+(a, x, k), \quad (81)$$

$$\delta_-(a, x, k) = \mathbf{1} - \sum_{n=0}^{k-1} \left( \hat{\Psi}_a^{(n)} \bar{\delta}_+^s(a, x, k-n) + \hat{\Lambda}_a^{(n)} \bar{\delta}_-(0, x, k-n) \right), \quad (82)$$

and for  $a > x$ , by

$$\delta_+(a, x, k) = \mathbf{0}, \quad \text{and} \quad \delta_-(a, x, k) = \mathbf{0}. \quad (83)$$

*Proof.* For  $k \geq 1$ ,  $x \geq 0$ , any  $i$ ,  $\bar{\delta}_i(0, x, k)$  is the probability that the process reaches level  $x$  before time  $\mathcal{Y}_k$ . Thus,

$$\bar{\delta}_+(0, x, k) = \sum_{n=0}^{k-1} \Lambda_x^{(n)} \mathbf{1} + \sum_{n=0}^{k-1} \Psi_x^{(n)} \sum_{l=0}^{k-n-1} \Upsilon^{(l)} \bar{\delta}_+(0, x, k-n-l). \quad (84)$$

To show this : the first term is the probability that the level reaches  $x$  without returning to level 0; the second term is the probability that the process first comes back in level 0 under taboo of level  $x$ , during the  $n$ -th stage, with probability  $\Psi_x^{(n)}$ , then the level stays in level 0 for  $l$  Exponential intervals, with probability  $\Upsilon^{(l)}$  at which time the process is in the initial situation, but with  $k-n-l$  stages left: this is given by the probability  $\bar{\delta}_+(0, x, k-n-l)$ . We reorganize the terms in equation (84) and obtain (77), as  $(I - \Psi_x^{(0)} \Upsilon^{(0)})$  is non-singular.

For  $k \geq 1$ , the probability vector  $\delta_+(0, x, k)$  is trivially the complement probability vector of (77), so we have Equation (78).



For  $k \geq 1$ , equation (79) is obtained by noting that for  $i \in \mathcal{S}_-$ , the complementary probability of  $\delta_-(0, x, k)$  is given by

$$\bar{\delta}_-(0, x, k) = \sum_{n=0}^{k-1} \Upsilon^{(n)} \bar{\delta}_+(0, x, k-n) :$$

it is the probability  $\Upsilon^{(n)}$  that the level first leaves 0 in one of the  $k$  Erlang stages, times the probability  $\bar{\delta}_+(0, x, k-n)$  that the level reaches the level  $x$ , in one of the  $k-n$  stages left. So, equation (79) is proved.

Take  $a > 0$ , and  $n \geq 1$ . For  $x \geq a$ ,  $\bar{\delta}_+(a, x, n)$  can be decomposed in a manner similar to (84) as follows,

$$\begin{aligned} \bar{\delta}_+(a, x, k) &= \sum_{n=0}^{k-1} \Lambda_{x-a}^{(n)} + \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \Psi_{x-a}^{(n)} \hat{\Lambda}_a^{(l)} \bar{\delta}_-(0, x, k-n-l) \\ &\quad + \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \Psi_{x-a}^{(n)} \hat{\Psi}_a^{(l)} \bar{\delta}_+(a, x, k-n-l). \end{aligned} \quad (85)$$

Indeed, we have three cases:

**Case 1.** The process avoids  $a$ : it reaches  $x$  in the  $n$ -th stage without returning to level  $a$ .

**Case 2.** The process returns to  $a$  and drops to 0: the process first comes back in level  $a$  with the taboo of level  $x$ , with probability  $\Psi_{x-a}^{(n)}$ , then it reaches level 0 in  $l$ , one of the  $k-n$  stages left, before returning to  $a$ , with probability  $\hat{\Lambda}_a^{(l)}$ , and finally the process reaches level  $x$ , starting from level 0, with probability  $\bar{\delta}_-(0, x, k-n-l)$ .

**Case 3.** The process returns twice to  $a$  avoiding  $x$  and 0 : first it comes back to level  $a$  from above in the  $n$ -th stage under taboo of level  $x$ , with probability  $\Psi_{x-a}^{(n)}$ , then it comes back to level  $a$  from below in the  $l$ -th stage, with the taboo of level 0, with probability  $\hat{\Psi}_a^{(l)}$  and finally the level reaches  $x$  in one of the  $k-n-l$  remaining stages, with probability  $\bar{\delta}_+(a, x, k-n-l)$ .

After algebraic manipulation of the terms in (85) we obtain (80) since  $(I - \Psi_{x-a}^{(0)} \hat{\Psi}_a^{(0)})$  is non-singular.

For  $k \geq 1$  and  $x > a$ ,  $\delta_+(a, x, k)$  given in Equation (81) is immediate.

To show (82), we note that the complementary probability vector of  $\delta_-(a, x, k)$ , can be decomposed as follows,

$$\bar{\delta}_-(a, x, k) = \sum_{n=0}^{k-1} \hat{\Psi}_a^{(n)} \bar{\delta}_+(a, x, k-n) + \sum_{n=0}^{k-1} \hat{\Lambda}_a^{(n)} \bar{\delta}_-(0, x, k-n). \quad (86)$$

To see this: the first term is the probability that the process first returns to the level  $a$  in stage  $l$ , before it reaches level 0, i.e.  $\hat{\Psi}_a^{(n)}$ , multiplied by the probability that the level reaches level  $x$  during one of the  $k - n$  stages left, i.e.  $\bar{\delta}_+(a, x, k - n)$ ; the second term is the probability that the process first reaches the level 0 before returning to level  $a$ , this is  $\hat{\Lambda}_a^{(n)}$ , multiplied by the probability that the process reaches level  $x$ , given that the initial level is 0, the phase is in  $\mathcal{S}_-$  and that there are  $k - n$  stages left, i.e.  $\bar{\delta}_-(0, x, k - n)$ .

If  $a > x$ , then obviously  $\delta(a, x, k) = 0$ , as it is not possible that the maximum of the process should be less or equal to  $x$  if the initial level is already greater than  $x$ . This gives (83).  $\square$

Note that equations (77) and (80) simplify a lot when  $k = 1$  because in this case, they contain sums on empty sets.

## 5.4 Joint distribution of $m_0(T)$ and $Z(T)$

Denote the joint distribution vector of  $m_0(\mathcal{Y}_k)$  and  $Z(\mathcal{Y}_k)$  by  $\mathcal{P}^{\rho, \mathbf{q}}(a, x, y, k)$ , which is the vector with components

$$\mathcal{P}_i^{\rho, \mathbf{q}}(a, x, y, k) = \mathbb{P}_i^a [m_0(\mathcal{Y}_k) \leq x, Z(\mathcal{Y}_k) \leq y],$$

where  $i \in \mathcal{S}$ ,  $1 \leq k \leq L$ ,  $a, x \geq 0$ .

**Theorem 5.7.** *The conditional joint distribution of  $m_0(\mathcal{Y}_k)$  and  $Z(\mathcal{Y}_k)$  given the initial level and the initial phase is given as follows:*

(a) If  $0 \leq a \leq x$ ,

$$\mathcal{P}_+^{\rho, \mathbf{q}}(a, x, y, k) = \mathbf{q}_+(a, y, k), \quad \text{and} \quad \mathcal{P}_-^{\rho, \mathbf{q}}(a, x, y, k) = \mathbf{q}_-(a, y, k). \quad (87)$$

(b) If  $0 \leq x < a$ ,

$$\mathcal{P}_+^{\rho, \mathbf{q}}(a, x, y, k) = \sum_{n=0}^{k-1} \Psi^{(n)} \sum_{l=0}^{k-n-1} \mathcal{P}_-^{\rho, \mathbf{q}}(a, x, y, k - n), \quad (88)$$

$$\mathcal{P}_-^{\rho, \mathbf{q}}(a, x, y, k) = \sum_{n=0}^{k-1} \mathbf{W}_{a-x}^{(n)} \mathbf{q}_-(x, y, k - n). \quad (89)$$

*Proof.* Equation (87) is easily justified:  $m_0(\mathcal{Y}_k)$  is at most equal to  $a$  so that if  $x \geq a \geq 0$ ,

$$\mathcal{P}_i^{\rho, q}(a, x, y, k) = \mathbb{P}_i^a [Z(\mathcal{Y}_k) \leq y] = q_i(a, y, k),$$

by definition, and is given in (58).

Take now  $0 \leq x < a$ . Equation (89) is interpreted as follows. Given a phase in  $\mathcal{S}_-$ , the process must reach down to  $x$  during the  $n$ -th exponential interval, for some  $n$ , and, being at level  $x$ , it has to be at most equal to  $y$  at the end of the remaining time. The probability of the first part of the trajectory is  $\mathbf{W}_{a-x}^{(n)}$  and the probability of the second part is  $\mathbf{q}_-(x, y, k - n)$ .

Equation (88) is found by noting that the process reaches level  $x$ , given the initial level  $a$  in a phase of  $\mathcal{S}_+$ , by first returning to level  $a$  in one of the  $k$  exponential stages, with probability  $\Psi^{(n)}$ ,  $n \in \{0, \dots, k - 1\}$ . Then the situation is the same as in the previous case.  $\square$

## 5.5 Joint distribution of $M_0(T)$ and $Z(T)$

Denote the joint distribution of  $M_0(\mathcal{Y}_k)$  and  $Z(\mathcal{Y}_k)$  by  $\mathcal{P}^{\delta, q}(a, x, y, k)$ , which is the vector with components

$$\mathcal{P}_i^{\delta, q}(a, x, y, k) = \mathbb{P}_i^a [M_0(\mathcal{Y}_k) \leq x, Z(\mathcal{Y}_k) \leq y], \quad (90)$$

where  $i \in \mathcal{S}$ ,  $1 \leq k \leq L$ ,  $a, x \geq 0$ . To simplify the notation in what follows, we define the vector of probabilities  $\mathcal{Q}(a, x, y, k)$  with components

$$\mathcal{Q}_i(a, x, y, k) = \mathbb{P}_i^a [M_0(\mathcal{Y}_k) > x, Z(\mathcal{Y}_k) \leq y]. \quad (91)$$

Of course, we have

$$\mathcal{P}^{\delta, q}(a, x, y, k) = \mathbf{q}(a, y, k) - \mathcal{Q}(a, x, y, k), \quad (92)$$

where  $\mathbf{q}(a, y, k)$  is given in the Theorem 5.3.

**Theorem 5.8.** *The joint distribution of  $M_0(\mathcal{Y}_k)$  and  $Z(\mathcal{Y}_k)$  given the initial level and the initial phase is given by (92) where  $\mathcal{Q}(a, x, y, k)$  is characterized as follows.*

(a) Take  $0 = a \leq x$  and  $1 \leq k \leq L$ .

$$\begin{aligned} \mathcal{Q}_+(0, x, y, k) &= \left( I - \Psi_x^{(0)} \Upsilon^{(0)} \right)^{-1} \left( \sum_{n=0}^{k-1} \Lambda_x^{(n)} \mathbf{q}_+(x, y, k-n) \right. \\ &\quad \left. + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \Psi_x^{(n)} \Upsilon^{(m)} \mathcal{Q}_+(0, x, y, k-m-n) \right), \end{aligned} \quad (93)$$

$$\mathcal{Q}_-(0, x, y, k) = \sum_{n=0}^{k-1} \Upsilon^{(n)} \mathcal{Q}_+(0, x, y, k-n), \quad (94)$$

(b) Take  $a > 0$ .

(i) Take  $x \geq a$ .

$$\begin{aligned} \mathcal{Q}_+(a, x, y, k) &= \left( I - \Psi_{x-a}^{(0)} \hat{\Psi}_a^{(0)} \right)^{-1} \left( \sum_{n=0}^{k-1} \Lambda_{x-a}^{(n)} \mathbf{q}_+(x, y, k-n) \right. \\ &\quad \left. + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \Psi_{x-a}^{(m)} \mathcal{L}(n, k-m) \right), \end{aligned} \quad (95)$$

$$\mathcal{Q}_-(a, x, y, k) = \sum_{n=0}^{k-1} \mathcal{L}(n, k), \quad (96)$$

where

$$\mathcal{L}(n, k) = \hat{\Psi}_a^{(n)} \mathcal{Q}_+(a, x, y, k-n) + \hat{\Lambda}_a^{(n)} \mathcal{Q}_-(0, x, y, k-n). \quad (97)$$

(ii) Take  $x < a$ .

$$\mathcal{Q}_+(a, x, y, k) = \mathbf{q}_+(a, y, k) \quad \text{and} \quad \mathcal{Q}_-(a, x, y, k) = \mathbf{q}_-(a, y, k). \quad (98)$$

*Proof.* Take  $a = 0$  and  $x \geq 0$ . Then,  $\mathcal{Q}_+(0, x, y, k)$  is as follows

$$\begin{aligned} \mathcal{Q}_+(0, x, y, k) &= \sum_{n=0}^{k-1} \Lambda_x^{(n)} \mathbf{q}_+(x, y, k-n) \\ &\quad + \sum_{\substack{0 \leq m, n \\ 1 \leq m+n \leq k-1}} \Psi_x^{(n)} \Upsilon^{(m)} \mathcal{Q}_+(0, x, y, k-m-n), \end{aligned} \quad (99)$$

for  $y \geq 0$ . Indeed, the first term, is the probability that the level reaches level  $x$  in stage  $n$  before returning to level 0; then one have the conditional probability that given that the process is in level  $x$  in a phase of  $\mathcal{S}_+$ , the process has to be less than  $y$  at time  $\mathcal{Y}_{k-n}$ . The second term is the probability that the level returns to 0 in a phase of  $\mathcal{S}_-$  before level  $x$  is reached, with probability  $\Psi_x^{(n)}$ , then it spends some time in level 0, with probability  $\Upsilon^{(m)}$ , and when the phase changes to  $\mathcal{S}_+$ , the situation is as the initial one, except that there are  $k - n - m$  stages left. After a reorganization of the terms of (99) we obtain (93). To prove (94) is obvious.

Take  $x \geq a > 0$ . If the initial phase is in  $\mathcal{S}_-$ , there are two ways to reach  $x$  before  $\mathcal{Y}_k$  and to be below  $y$  at time  $\mathcal{Y}_k$ , which are combined in  $\mathcal{L}(\cdot, \cdot)$  given in (97). The first is to return from below to level  $a$  in a phase of  $\mathcal{S}_+$  before time  $\mathcal{Y}_k$  and before reaching level 0, with probability  $\hat{\Psi}_a^{(n)}$ ; then the situation is as if the process starts in  $\mathcal{S}_+$  but with  $k - n$  stages left only, thus we have to multiply this by  $\mathcal{Q}_+(a, x, y, k - n)$ . The second way is to reach level 0 before returning to  $a$ , with probability  $\hat{\Lambda}_a^{(n)}$  and then again, the situation is as if the process starts in a phase of  $\mathcal{S}_-$ , in level 0, this probability is given by  $\mathcal{Q}_-(0, x, y, k - n)$ .

The proof of (95) is nearly identical to the proof of (93). The first term in the big bracket is equal to the first term in (99) starting from  $a$  instead of 0; but the second one is slightly different because when the level first come back to the initial level  $a$ , before crossing level  $x$ , with probability  $\Psi_{x-a}^{(m)}$ , it does so in a phase of  $\mathcal{S}_-$ . Then there are two possibilities which are given in  $\mathcal{L}(n, k - m)$  as there are  $k - m$  stages left. A simple algebraic manipulation leads then to equation (95).

The case  $0 \leq x < a$  is obvious.  $\square$

Remark that here again, we have important simplifications when  $k = 1$  for equations (93) and (95) as they contain sums on empty sets.

## 6 Numerical illustration

We illustrate our results with the following example: the value of some asset normally evolves in one of two environments: it increases in environment 1 and decreases otherwise. Occasionally, the rates of variation become much higher, for short periods

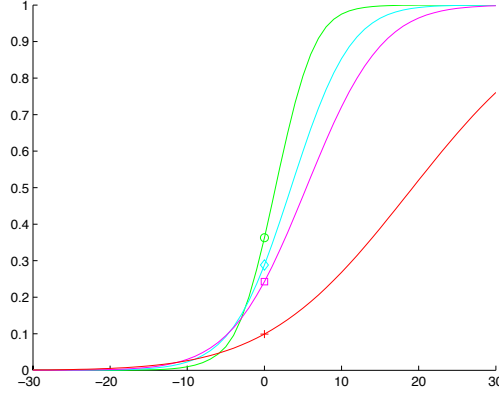


Figure 1: Distribution function of the level at different maturities, where  $L = 30$  is constant. The curve with the symbol  $\circ$  for  $T = 5$ ,  $\diamond$  for  $T = 10$ ,  $\square$  for  $T = 15$  and  $+$  for  $T = 50$ .

of time, indicating unusually high activity. Precisely, the generator is

$$A = \begin{array}{c} \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{array} \end{array} \left[ \begin{array}{cc|cc} -\lambda - \omega & \lambda & p\omega & (1-p)\omega \\ \lambda & -\lambda - \omega & p\omega & (1-p)\omega \\ \hline \mu & 0 & -\mu - \beta & \beta \\ 0 & \mu & \beta & -\mu - \beta \end{array} \right].$$

Phases 1 and 2 correspond to the calm environment, (CE) with  $c_1 = 2$  and  $c_2 = -1$ , to reflect the fact that the value of the asset is generally increasing. Phases 3 and 4 correspond to the excited environment (EE), with  $c_3 = 10$ ,  $c_4 = -10$ . The unit of time is one week, and we take  $\lambda = 1$ . The parameter  $\omega$  is equal to 0.25 and  $\mu = 1$  so that the process remains for four weeks in the average in the CE before moving to the EE where it remains for one week, on average. Finally,  $\beta = 7$ , so that during the EE, switching from increase to decrease occurs every day.

We take  $\varphi(0) = 2$  so that the level starts decreasing at the rate  $-1$  and we assume that  $X(0) = 0$ .

The distribution for  $L$  fixed and different maturities  $T$  is depicted on Figure 1. Globally we see that  $X(T)$  increases when  $T$  increases. This is due to the fact that  $c_1 = 2$  and  $c_2 = -1$  so that the stationary drift  $\alpha C \mathbf{1}$  is slightly positive, where  $\alpha$  is the stationary probability vector of  $A$  and  $C$  is the rate matrix.

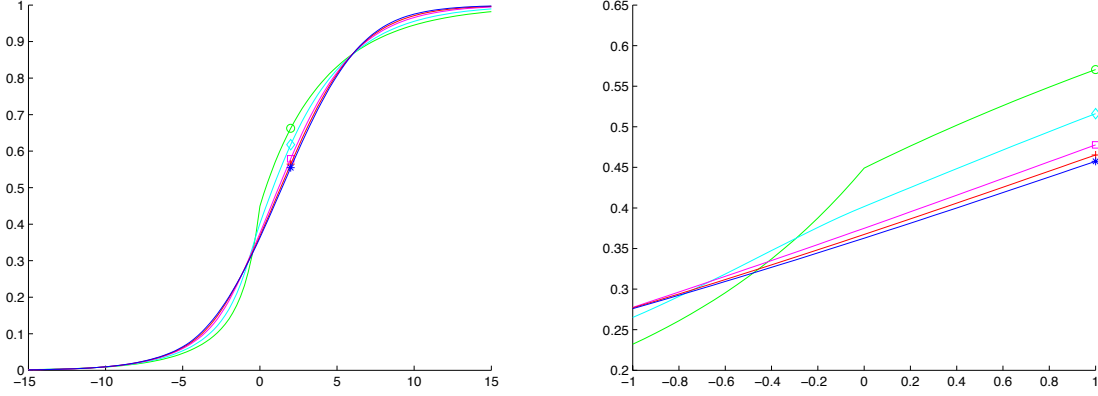


Figure 2: Distribution function of the level at maturity, where the Erlang approximating time has mean  $\theta = 10$ , conditionally given  $X(0) = 0$  and  $\varphi(0) = 2$ . The curve with the symbol  $\circ$  is for the parameter  $L = 1$ ,  $\diamond$  for  $L = 2$ ,  $\square$  for  $L = 5$ ,  $+$  for  $L = 10$  and  $*$  for  $L = 30$ .

We plot on Figure 2 the distribution function of the level at Erlang maturities  $T \sim \text{Erl}(L/\theta, L)$  for  $\theta = 10$  and  $L = 1, 2, 5, 10$  and  $30$ . In general, the curves are quite smooth, with the exception of the density for  $L = 1$  if  $\phi(0)$  is in  $\mathcal{S}_-$ : we display in greater detail the functions in a small interval around 0 and we clearly see that the curve for  $L = 1$  has a different aspect at  $x = 0$ . We write

$$\mathbf{r}'_-(0^-, 1) = \lim_{x \uparrow 0} \frac{\partial}{\partial x} \mathbf{r}_-(x, 1) \quad \text{and} \quad \mathbf{r}'_-(0^+, 1) = \lim_{x \downarrow 0} \frac{\partial}{\partial x} \mathbf{r}_-(x, 1).$$

It appears from Figure 2 that the density is discontinuous there with  $\mathbf{r}'_-(0^-, 1) > \mathbf{r}'_-(0^+, 1)$ . This we interpret as follows. In general, the value for  $X(T)$  is the result of multiple changes, which results in smooth curves. If  $L = 1$  and  $\phi(0) = i \in \mathcal{S}_-$ , however there is a small probability, of the order of  $\nu h$  that the process is absorbed almost at once, in a small interval of length  $h$ . If that happens, the fluid level is  $c_i h < 0$  and close to 0. This explains why the difference between the limits of the density from below and from above is  $\nu/|c_i|$ , as we now show precisely. On the one hand,

$$\begin{aligned} \mathbf{r}'_-(0^-, 1) &= \lim_{x \uparrow 0} \frac{\partial}{\partial x} \mathbf{W}_{|x|} \hat{\mathbf{h}}(1), \\ &= -\mathbf{U} \hat{\mathbf{h}}(1), \\ &= -|C_-^{-1}| (A_{--} - \nu I) \hat{\mathbf{h}}(1) - |C_-^{-1}| A_{-+} (\mathbf{1} - \mathbf{h}(1)), \end{aligned}$$

by Proposition 4.2, and on the other hand we have that

$$\begin{aligned}
\mathbf{r}'_-(0^+, 1) &= \lim_{x \downarrow 0} \frac{\partial}{\partial x} (\mathbf{1} - \hat{\Psi} \hat{\mathbf{W}}_{|x|} \mathbf{h}(1)), \\
&= -\hat{\Psi} \hat{\mathbf{U}} \mathbf{h}(1), \\
&= (-\hat{\Psi} C_+^{-1} (A_{++} - \nu I) - \hat{\Psi} C_+^{-1} A_{+-} \hat{\Psi}) \mathbf{h}(1), \\
&= \left( |C_-^{-1}| (A_{--} - \nu I) \hat{\Psi} + |C_-^{-1}| A_{-+} \right) \mathbf{h}(1),
\end{aligned}$$

by Theorem 3.2. By Proposition 4.2 we obtain

$$\mathbf{r}'_-(0^+, 1) = |C_-^{-1}| (A_{--} - \nu I) (\mathbf{1} - \hat{\mathbf{h}}(1)) + |C_-^{-1}| A_{-+} \mathbf{h}(1),$$

so that

$$\begin{aligned}
\mathbf{r}'_-(0^-, 1) - \mathbf{r}'_-(0^+, 1) &= -|C_-^{-1}| (A_{--} - \nu I) \mathbf{1} - |C_-^{-1}| A_{-+} \mathbf{1} \\
&= \nu |C_-^{-1}| \mathbf{1},
\end{aligned}$$

as  $Q\mathbf{1} = 0$ , or

$$r'_i(0^-, 1) = r'_i(0^+, 1) + \frac{\nu}{|c_i|},$$

for any  $i \in \mathcal{S}_-$ .

### Remark 6.1. Computational issues

In this example, the total number  $Lm$  of phases is sufficiently small and we have computed the matrix  $\mathbf{W}_x = \exp(\mathbf{U}x)$  by using the function `expm` from MATLAB [9]. When  $Lm$  is large, we need to reduce the cost of computing  $\mathbf{W}_x$  which is  $O(L^3 m^3)$  (see Moler and Van Loan [11]) unless one uses structural properties of  $\mathbf{U}$ .

One approach is to apply Algorithm 1 of Xue and Ye [16] which is designed for the computation of general exponentials of essentially non-negative matrices entry-wise to high relative accuracy.

In addition to computing  $\mathbf{W}_x$  fast and accurately, we wish to compute the blocks  $\mathbf{W}_x^{(0)}, \dots, \mathbf{W}_x^{(n)}$  only, as they completely specify  $\mathbf{W}_x$ , and this allows to fully benefit from the decomposition approach followed in this paper. Efficient algorithms to that effect are developed in Bini *et al.* [5]. One method considered there consists in exploiting the Toeplitz structure to specialize the shifting and Taylor series method of [16].



Another highly efficient method, to reduce the computation cost is based on the block-circulant matrix method: we define the matrix

$$\mathbf{V}_\epsilon = \begin{bmatrix} \mathbf{U}^{(0)} & \mathbf{U}^{(1)} & \mathbf{U}^{(2)} & & \mathbf{U}^{(L-1)} \\ \epsilon \mathbf{U}^{(L-1)} & \mathbf{U}^{(0)} & \mathbf{U}^{(1)} & & \mathbf{U}^{(L-2)} \\ \epsilon \mathbf{U}^{(L-2)} & \epsilon \mathbf{U}^{(L-1)} & \mathbf{U}^{(0)} & & \mathbf{U}^{(L-3)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \epsilon \mathbf{U}^{(1)} & \dots & & \epsilon \mathbf{U}^{(L-1)} & \mathbf{U}^{(0)} \end{bmatrix}.$$

where  $\epsilon$  is some small number. This is a block  $\epsilon$ -circulant matrix which may be block-diagonalized by Fast Fourier Transforms techniques, so that the computation of  $\exp(\mathbf{V}_\epsilon x)$  is reduced to  $O(m^2 L \log_2 L)$  the computation of  $L$  exponentials of matrices of order  $m$ , and serves as a close approximation of  $\mathbf{W}_x$ .

A third approach investigated in Bini *et al.* [5] is as follows: one defines the matrices  $S^{(0)}, \dots, S^{(K-1)}$ , for  $K \geq L$ , with

$$S^{(i)} = \begin{cases} \mathbf{U}^{(i)}, & \text{for } i \leq L-1, \\ 0, & \text{for } L \leq i \leq K-1, \end{cases}$$

and the block-circulant matrix

$$S = \begin{bmatrix} S^{(0)} & S^{(1)} & S^{(2)} & \dots & S^{(K-1)} \\ S^{(K-1)} & S^{(0)} & S^{(1)} & \dots & S^{(K-2)} \\ S^{(K-2)} & S^{(K-1)} & S^{(0)} & \dots & S^{(K-3)} \\ \vdots & & \ddots & \ddots & \vdots \\ S^{(1)} & \dots & & S^{(K-1)} & S^{(0)} \end{bmatrix}.$$

The matrix  $\exp(Sx)$  may be efficiently computed with a complexity  $O(m^2 K \log_2 K)$  plus the cost of  $K$  exponentials of matrices of order  $m$ , by means of FFT techniques and, for  $K$  large enough, the blocks  $[\exp(Sx)]_{0,i}$ , for  $0 \leq i \leq L-1$ , constitute a good approximation of  $\mathbf{W}_x^{(0)}, \dots, \mathbf{W}_x^{(L-1)}$ .

Full details about these three approximation methods are given in Bini *et al.* [5].

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